Tools for finding numerical algorithms

Your answer may be off in the exponent by one, but if it is more than that, please speak to the instructor or a teaching assistant or your peers. You should use a program like C++. If you find this programming to be frustrating, please consider visiting

https://replit.com/@dwharder/3-Tools-for-finding-numerical-algorithms,

but not until after you try this on your own, please.

1. For what power of two is the approximation of the derivative of tan(*x*) most accurate when approximating the derivative at $x = 0.001$ (one milliradian) or $x = 1.57$? Use the approximation $(\tan(x + h) - \tan(x))/h$.

Answer: When $x = 0.001$, it appears to be most accurate when $h = 2^{-26}$. When $x = 1.57$, it appears to be most accurate when $h = 2^{-37}$.

2. We have this approximation of the derivative:

$$f^{(1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) \approx \frac{f(x+h) - f(x)}{h} .$$

Now, the second derivative is:

$$f^{(2)}(x) = \frac{\mathrm{d}^2}{\mathrm{d}^2 x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right).$$

Now, the entry in the brackets is $f^{(1)}(x)$, so we have

$$f^{(2)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(1)}(x) \approx \frac{f^{(1)}(x+h) - f^{(1)}(x)}{h}.$$

Now, if $f^{(1)}(x) \approx \frac{f(x+h) - f(x)}{h}$ then $f^{(1)}(x+h) \approx \frac{f((x+h)+h) - f(x+h)}{h}$, so substituting these into the previous equation, we get

$$f^{(2)}(x) \approx \frac{f^{(1)}(x+h) - f^{(1)}(x)}{h} \approx \frac{\dfrac{f((x+h)+h) - f(x+h)}{h} - \dfrac{f(x+h) - f(x)}{h}}{h}.$$

With a little bit of algebra, we see that

$$f^{(2)}(x) \approx \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}.$$

For what value of *h* is this most accurate for approximating the second derivative of sin(*x*) at $x = 1$?

Answer: It seems to have a minimum error when $h = 2^{-17}$ and the absolute error here is approximately 5.455×10^{-6}.

3. Here is an alternative formula for the second derivative, which we will offer without proof:

$$f^{(2)}(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

For which value of h is this most accurate for approximating the second derivative of sin(x) at $x = 1$.

Answer: It seems to have a minimum error when $h = 2^{-13}$ and the absolute error here is approximately 1.797×10^{-9}.

4. Which formula is a better approximation of the second derivative, at least with the evidence presented?

Answer: It appears the formula in Question 3 is more accurate.